Building up to Lorentzian Causality Theory

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DRP 2020

16 January 2020

Lorentzian causality theory is the subfield of mathematical relativity concerned with the study of causal relation between points on a Lorentzian manifold. Lorentzian manifolds are objects that we use to model space-time. In this talk, we will focus on discussing the foundations of causality theory (which consists of differential geometry and special relativity) and providing a brief introduction to Lorentzian causality theory.

Introduction: Historical Development

1. Field of mathematical relativity concerned with causal relations between points on a Lorentzian manifold.
2. "Gravitational Collapse and Space-time singularities":
   1. Predicted conditions under which a space-time singularity would form.
   2. Initiated Lorentzian causality theory.
3. 1966: Hawking applied Penrose's argument to the universe as a whole to predict that the universe began as a singularity.
   1. Sparked the development of mathematical relativity.
4. Physicists more interested in studying black holes.
   1. Could shed light on the unification of gravity with the other three forces of nature.

For our purposes, the precise definition of a topological space is not important. What is important to understand are the definitions of Hausdorff and second-countable spaces.

Definition (Hausdorff)

A topological space M is said to be **Hausdorff** if $\forall\ x,\ y \in M$, there exists open sets Ω_1 and Ω_2 such that $x \in \Omega_1$ and $y \in \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$.

Definition (Second-Countable)

A topological space M is said to be **second-countable** if there exists a countable collection of open sets of M, $\{\Omega_n\}$, such that for any open set $O \subseteq M$, O can be written as a union of elements Ω in $\{\Omega_n\}$.

Definition (Manifold)

A **manifold** is a topological space M with the following property: if $x \in M$, then there exists a neighbourhood Ω of x and some $n \in \mathbb{Z}$ such that Ω is homeomorphic to $\mathbb{R}^n$.

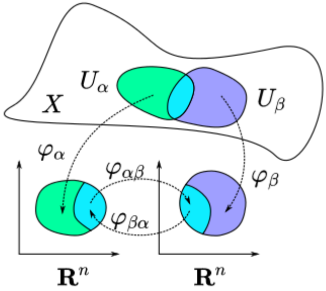

Figure: Credit: Manifolds: A Gentle Introduction

Definition (Tangent Space)

Informally, let M be a differentiable manifold. Then, the **tangent space** of a manifold M at the point p, denoted $T_p(M)$, is a vector space attached to p containing all the possible ways in which one can tangentially pass through p.

Definition (Convex Cone)

Let V be a vector space. Then, a **convex cone** C is a subset of V that is closed under positive linear combinations; that is, $\forall\ \alpha$, β positive, $x, y \in C \Rightarrow \alpha x + \beta y$.

Special Relativity is the branch of physics concerned with the relationship between space and time. Proposed by Albert Einstein in 1905 in response to the contradiction between Newtonian mechanics, Maxwell's theory of electromagnetism, and Newtonian physics. Principles of special relativity:

1. Speed of light c in a vacuum is the same for all reference frames.
2. Laws of physics are invariant under changes in reference frames.

Minkowski space models space-time. It's a four-dimensional manifold. Lorentz transformation is used to shift reference frames from one frame to another...

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

γ is called the **Lorentz factor**. Special relativity is only suitable for studying reference frames moving at velocities comparable to the speed of light.

Interested: in quantities that remain invariant under Lorentz transformations. The **invariant interval** is defined as:

$$\Delta s^2 := c^2t^2 - (x^2 + y^2 + z^2) \tag{1}$$

Invariant intervals ⇒ **Minkowski space-time cone** ⇒ space-like, time-like, and light-like separation.

Special Relativity

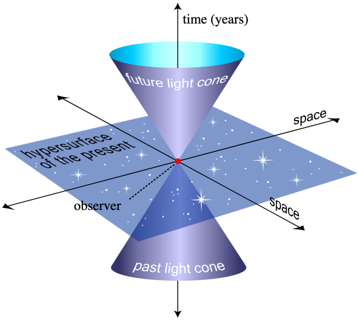

Figure: Minkowski light-cone; credits to Wikipedia

Points on the light cone satisfy $\Delta s^2 = 0$. This light cone is time-oriented since the top component has been labelled as “future” and the bottom component has been labelled as “past.”

Let A denote the origin...

1. If $c^2t^2 - (x^2 + y^2 + z^2) > 0 \Rightarrow A$ and B are **space-like separated**.
2. If $c^2t^2 - (x^2 + y^2 + z^2) = 0 \Rightarrow A$ and B are **light-like separated** if $B \neq A$ and **null-separated** if $B = A$.
3. If $c^2t^2 - (x^2 + y^2 + z^2) < 0$, we say that A and B are **time-like separated**.

Events that are either time-like or light-like separated are said to be **causally separated**. Since these definitions are formulated in terms of the invariant interval, these notions are invariant under the Lorentz transformation.

Talk Outline...

1. First Fundamental Form
2. Introduce some basic notions and definitions from causality theory

Motivation: In $\mathbb{R}^3$, if we wish to describe metric properties such as lengths of curves or areas of regions, we have the standard inner product $\langle \cdot, \cdot \rangle$. If we want to discuss metric properties on a surface $S \subseteq \mathbb{R}^3$ without referring to the ambient space, we need the **first fundamental form**

1. Captures how a manifold M inherits the inner product from its ambient space.

Definition (First Fundamental Form)

Let $S \subseteq \mathbb{R}^3$ be a regular surface. Let $w \in S$. Then, the quadratic form

$$I_p(w) := \langle w, w \rangle = ||w||^2 \geq 0 \tag{2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of the ambient space. This form is called the **first fundamental form** of S at the point p.

More enlightening way to express the first fundamental form is in terms of a **metric tensor**. It generalises the inner product of $\mathbb{R}^n$ to differentiable manifolds. Let S be parametrised by $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$, $(a, b) \mapsto (x, y, z)$, then the first fundamental form can be expressed as:

$$I_p(w) := w^2 \begin{bmatrix} E(a,b) & F(a,b) \\ F(a,b) & G(a,b) \end{bmatrix} w \tag{3}$$

The matrix is called a **metric tensor**.

Observe: this means the inner product depends on the location $p \in S$.

Signature of a metric: number of positive, negative, and zero eigenvalues of the metric tensor.

A Brief Introduction to Lorentzian Causality Theory: Basic Definitions and the Setup

Lorentzian causality theory studies the following differential inclusion:

$$x \mapsto C_x \subseteq T_x(M) \setminus \{0\} \tag{4}$$

These are convex cones in the tangent space of M at the point x.

1. **Future causal cone** at x:

$$C_x := \{y \in T_x(M) \setminus \{0\} \mid g(y, y) \leq 0\}$$

2. **Minkowski metric tensor**:

$$\eta := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1. **Observer space**: the set of all future-directed vectors in V such that $g(v,v) = -1$. Generates a **hyperboloid** in $\mathbb{R}^n$.
2. A **Lorentz map** is an endomorphism: $\Lambda : V \to V$ that ensures g is invariant:

$$\forall\ v, w \in V,\ \ g(\Lambda(v), \Lambda(w)) = g(v, w)$$

Two theorems from Causality Theory

Let (V, g) be a Minkowski space and let $C \subseteq V$ be the future causal cone. Define: $F : C \to [0, \infty[$

$$F(v) := \sqrt{-g(v, v)}$$

Theorem (Reverse Cauchy-Schwartz Inequality)

Let $v_1, v_2 \in C$.

$$-g(v_1, v_2) \leq F(v_1)F(v_2)$$

Where equality holds $\iff$ v_1 and v_2 are proportional.

Theorem (Reverse Triangle Inequality)

Let $v_1, v_2 \in C$.

$$F(v_1 + v_2) \geq F(v_1) + F(v_2)$$

Motivation: time-orientable is important since we define **space-time** to be a non-compact, time-oriented, smooth Lorentzian manifold.

1. **Connected space:** M is connected if we *cannot* find open, disjoint sets A, B such that $M = A \cup B$.
2. **Compact space:** M is compact if every open cover of M admits finite subcover.

Definition (Time-Orientable)

Let M be a Lorentzian manifold. We say that M is **time-orientable** if $\forall x \in M$, we can make a choice for the future cone of $(T_x(M), g_x)$ so that the choice varies continuously with x along M.

If a manifold M is not time-orientable, then it admits a time-orientable covering.

Let $p_0 \in M$ be a reference point. Consider the following set...

$$\mathcal{F} := \{(p, \gamma) \mid p \in M,\ \gamma \ \text{ continuous curve from } p_0 \text{ to } p\}$$

Define the following equivalence relation: $\mathcal{F} : (p, \gamma) \sim (p', \gamma')$ if

1. $p = p'$
2. Time-like vectors at p that are continuous moved from p to p_0 along γ, then back to $p = p'$ along γ', do not change their time-direction.

The manifold that is generated by taking all possible equivalence classes based on the rule above is called a **Lorentzian covering** of M.

Cone distributions and conformal invariance

Motivation: investigates the relation between different metrics on the same manifolds and how this relates to volume forms.

1. **Conformal mapping**: map that preserves angles but not necessarily lengths.

Theorem

Let V be a $(n+1)$-dimensional vector space, and let g and $\overline{g}$ be two Lorentzian bilinear forms on V. Then, g and $\overline{g}$ induce the same double cone of causal vectors if and only if there exists $\Omega^2 \in \,]0, \infty[$ such that $\overline{g} = \Omega^2 g$, that is, g and $\overline{g}$ are conformally related.

Applications: used to determine when two space-times are based on the same manifold M but with two different metrics g and $\overline{g}$ induce the same causal cones. (M, g) and $(M, \overline{g})$ share the same causal cones $\iff$ the metrics g and $\overline{g}$ are conformally-related.

Volume Forms

Metrics induce **volume forms**:

$$\mu(X_0, X_1, ..., X_N) := \sqrt{|\det g(X_i, X_j)|}$$

Volume forms ⇒ measures ⇒ integration & notion of size in general.

Space-times based on the same oriented manifold with the same causal cones share the same volume form if and only if they are actually the same space-time.

Extend the notions of time-like, space-like, and light-like separation to curves.

Definition

Let $x : I \to M$, $t \mapsto x(t)$, $I \subseteq \mathbb{R}$ be a piecewise C^1 curve. Then, x is said to be **causal**, **light-like**, or **space-like** if its tangent vector has the corresponding behaviour at every point on the curve.

We can discuss causal relations between points based on the character of the curves that connect them. Namely,

Definition (Causal Relations and Chronological Relations)

A **causal relation** is defined as:

$$J := \{(p,q) \in M \mid \exists \text{ causal curve connecting } p \text{ to } q \text{ or if } p = q\ \}$$

A **chronological relation** is defined as:

$$I := \{(p,q) \in M \mid \exists \text{ a time-like connecting } p \text{ to } q \text{ or if } p = q\ \}$$

Observe: the causal relation J is not necessarily closed.

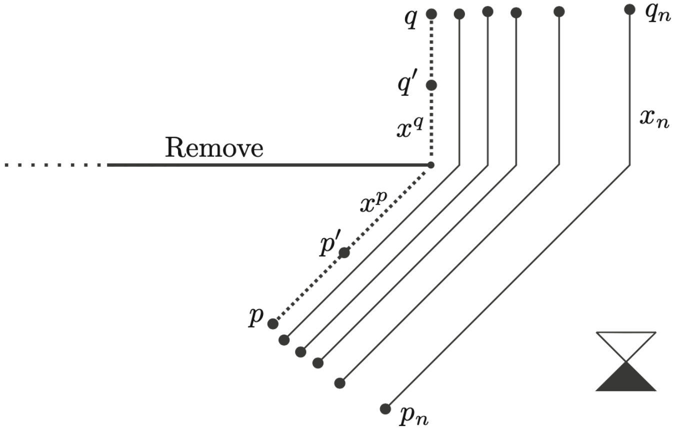

Figure: The construction of a sequence of points in J that converges to a point not in J (Source: [Min19]).

Future Work and Acknowledgements

This DRP project only scratched the surface of causality theory and special relativity. So, a natural progression of this paper is to continue reading the primary text that this paper drew from.

I would like to thank Vladmir Sicca Goncalves for the time he spent during the semester (and during the winter holidays!) for mentoring me and answering all my questions. I would also like to thank the DRP committee for organising this program. I learned a lot from this amazing experience.

Thank you for listening!

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