Spherical Harmonics
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Abstract

The purpose of this document is to build up to the derivation of the spherical harmonics on the 2-sphere and 3-sphere. To do this, we'll first provide some preliminary Functional Analysis definitions and results, solve the eigenvalue problem on easier domains such as on the interval $[0, \ell]$ and the rectangle $[0, \ell] \times [0, m]$, and then finally we will work out expressions for the spherical harmonics on the n-sphere for $n = 2$ and $n = 3$. We will collect all the key results at the end of the document for easy reference under the "Summary Tables for Quick Reference" section.

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1. Introduction and Motivation

We'll first recall that the **Fourier Series** are a set of functions used to represent functions on the circle S^1. Moreover, any "nice enough" function can be approximated by its Fourier Series. Recall from PDEs what a Fourier Series is:

Definition 1 (Fourier Series of f)**.** Let $f : \Omega \to \mathbb{R}$. We define the **Fourier Series** on $[-L, L]$ by:

$$FS[f(x)] := \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

where the Fourier Coefficients are obtained by taking inner products:

$$A_n := \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \tag{2}$$

$$B_n := \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{3}$$

$$\frac{1}{2}A_0 := \int_{-L}^{L} f(x)dx \tag{4}$$

For a nice enough function, the Fourier Series is a very good representation. In particular, one has

Theorem 1 (Uniform Convergence of Fourier Series)**.** Let $f \in C^2(]-L, L[)$. Then:

$$FS[f(x)] \to f(x) \tag{5}$$

uniformly in $]-L, L[$.

The convergence is pointwise if $f \in C^1(]-L, L[)$. We analogously want to represent functions on the 2-sphere and 3-sphere (or more generally, the n-sphere). Fourier series on the n-sphere are called **spherical harmonics**.

Spherical harmonics arise when one studies the so-called spectrum of the Laplace operator. Let's go back to linear algebra. For a matrix $\mathbf{A}$, one can study the set of eigenvalues of $\mathbf{A}$. The eigenvalues of $\mathbf{A}$ provide us with information about the nature of the linear transformation encoded by $\mathbf{A}$. Matrices only work for maps between finite-dimensional spaces; one can generalise this to any operator. An **operator** is a map acting on elements of a space to provide either elements of another space or the space itself. In this document, we will be focusing on linear operators.

Definition 2 (Linear Operator)**.** Let K be a field, and let U, V be vector spaces over the field K. A mapping $A : U \to V$ is said to be a **linear operator** if $\forall x, y \in U$, $\alpha, \beta \in K$, one has

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y) \tag{6}$$

Before proceeding, for the sake of completion, we will first provide some key results and definitions. They are taken straight out of Royden and Fitzpatrick's real analysis textbook.

Definition 3 (Inner Product)**.** Let H be a normed vector space. A function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ is called an **inner product** on H if $\forall x_1, x_2, x \in H$, $y \in H$, $\alpha, \beta \in \mathbb{R}$, one has:

(1) $\langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$ (We call this sesqui-linearity, or linearity in the first slot).
(2) $\langle x, y\rangle = \langle y, x\rangle$
(3) $\langle x, x\rangle = 0 \iff x = 0$.

$(H, \langle \cdot, \cdot \rangle)$ is called an **inner product space**.

Definition 4 (Hilbert Space)**.** An inner product space $\mathcal{H}$ is a **Hilbert Space** if its a Banach space with respect to the norm induced by the inner product.

Let $u \in \mathcal{H}$ be arbitrary, $u \neq 0$. Then, we say that u is an **eigenvector** or **eigenfunction** of the operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ if there exists $\lambda \in \mathbb{R}$ such that $T(u) = \lambda u$. We call λ the **eigenvalue** associated with the eigenvector or eigenfunction u. Finally, we'll define the spectrum of an operator

Definition 5 (Spectrum)**.** Let $A : X \to X$ be a bounded linear operator in a normed space X. The **spectrum** $\sigma(A)$ is the set $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not invertible.

For finite-dimensional spaces X, the spectrum of an operator is simply its eigenvalues. Things get a little more hectic with infinite-dimensional spaces, which is what we'll be working with. We are interested in studying the spectrum of the Laplacian. Before doing this, we will first provide some motivation. In particular, we will briefly discuss why one should be interested in studying the Laplacian.

1.1. Why do we care about the Laplacian?

We will first define the Laplacian – first in $\mathbb{R}^n$ and then we will generalise it to an arbitrary Riemannian manifold (M, g).

Definition 6 (Laplacian)**.** Let $\Omega \subseteq \mathbb{R}^n$ be connected. Then, the **Laplacian** is an operator acting on $C^\infty(\Omega)$ by summing the second partial derivatives:

$$\Delta \varphi := -\sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2} \tag{7}$$

For an arbitrary Riemannian manifold (M, g), the **Laplacian operator** is defined as:

$$\Delta_g : C^\infty(M) \to C^\infty(M)$$
$$\Delta_g := -\mathrm{div}_g \circ \nabla_g$$

(the subscript g's denote the operator with respect to that metric).

If one parametrises (M, g), then in local coordinates $(x_1, ..., x_n)$ one has the following expression for the Laplacian Δ_g:

$$\Delta_g = -\frac{1}{\sqrt{|\det(g)|}} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{|\det(g)|} \frac{\partial}{\partial x_j} \right) \tag{8}$$

where g^{ij} denotes the inverse of the metric g_{ij}. The Laplacian is featured in several physical problems, including:

1.2. Steady State Fluid Flow

Consider a fluid moving in $\mathbb{R}^3$. One often times one wants to study the evolution of the velocity of the fluid $v(x_1, x_2, x_3, t)$. To do so, we wish to study the steady states of the fluid. In this case, the velocity of the fluid would be independent of time, and so we write that for some $c_0 \in \mathbb{R}$:

$$v(x_1, x_2, x_3, t) = v(x_1, x_2, x_3, c_0) \tag{9}$$

Now recall from vector calculus the notions of irrotational and incompressible fluids. We say that a fluid is **irrotational** if $\mathrm{curl}(\mathbf{V}) = 0$. This implies that there exists a velocity potential u for $\mathbf{V}$, u.e.:

$$\mathbf{V} = -\nabla u \tag{10}$$

We say that a fluid is **incompressible** if $\mathrm{div}(\mathbf{V}) = 0$. If one has an irrotational incompressible fluid, then the velocity potential u satisfies **Laplace's Equation**:

$$\Delta u = 0 \tag{11}$$

This PDE is the prototypical example of an "elliptic PDE." We say that a function for which (11) holds is **harmonic**.

1.3. Heat Diffusion

Recall from PDEs the heat or diffusion equation. For a region $\Omega \subseteq \mathbb{R}^n$, if one wants to study the propagation of heat through Ω, one needs to study the following PDE:

$$\Delta u(x,t) = -\frac{1}{c}\frac{\partial}{\partial t}u(x,t) \tag{12}$$

where $u(x,t)$ models the temperature at the point $x \in \mathbb{R}^n$ at time $t \in [0, \infty[$. c is a constant which encodes the material's conductivity. The Heat Equation is the prototypical example of a "parabolic PDE."

1.4. Wave Propagation

This equation is very familiar to us. It's given by

$$\Delta u(x,t) = -\frac{1}{c}\frac{\partial^2}{\partial t^2} \tag{13}$$

where $\sqrt{c}$ is the speed of the wave in the fluid and $u(x,t)$ is the height of the wave at location $x \in \mathbb{R}^n$ and at time $t \in [0, \infty[$. The wave equation is the prototypical example of a "hyperbolic PDE."

1.5. Schrödinger's Equation

Finally, Schrödinger's Equation models the movement of a quantum particle in a domain $\Omega \subseteq \mathbb{R}^n$. It's given by:

$$\frac{\hbar^2}{2m}\Delta u(x,t) = i\hbar\frac{\partial}{\partial t}u(x,t) \tag{14}$$

where $\hbar$ is Planck's constant and m is the mass of the free particle. One can normalise $u(x,t)$ such that $||u(\cdot,t)||_{L^2(\Omega)} = 1$, which gives us a probabilistic interpretation:

$$\mathbb{P}[A; t_0] = \int_A |u(x, t_0)|dx \tag{15}$$

$\mathbb{P}[A; t_0]$ is the probability of finding the particle in a region $A \subseteq \Omega$ at a given time t_0.

The Laplacian in particular is vital to studying any physical process governed by laws that are independent of position and direction. Mathematically, this idea can be expressed as follows: let S be an operator that commutes with translations and rotations. Then, there exist coefficients $a_1, ..., x_n$ such that

$$S = \sum_{j=1}^{n} a_j \Delta^j \tag{16}$$

2. Solving the Eigenvalue Problem

Spherical harmonics appear when one studies the so-called **eigenvalue problem** or the **Helmholtz equation**:

$$\Delta\varphi = \lambda\varphi \tag{17}$$

One is interested in solving the problem above since it arises when one tries to solve the heat, wave, and Schrödinger equation using a technique from PDEs called "separation of variables." As an example, we will do it for the heat equation. This method is inspired by the Stone-Weierstrass theorem from real analysis.

2.1. Analysis Results

Before giving the theorem, we first need two definitions.

Definition 7 (Algebra)**.** A vector subspace $\mathcal{A} \leq C(X)$ is called an **algebra** if, for $f, g \in \mathcal{A}$, one has that $f \cdot g \in \mathcal{A}$.

Definition 8 (Separate Points)**.** A collection $\mathcal{A}$ of real-valued functions on X is said to **separate points** in X provided that $\forall$ distinct points $u, v \in X$, $\exists f \in \mathcal{A}$ such that $f(u) \neq f(v)$.

Theorem 2 (Stone-Weierstrass Theorem)**.** Let X be a compact Hausdorff space. Suppose that $\mathcal{A}$ is an algebra of continuous real-valued functions on X that separates points in X and contains the constant functions. Then, $\mathcal{A}$ is dense in $C(X)$.

2.2. Separation of Variables

Now we can attempt to solve (17) by separation of variables. To do son, one makes the following "Ansatz": assume that there exists a solution $u(x,t)$ of the form

$$u(x,t) = \alpha(t)\varphi(x) \tag{18}$$

Substituting this into the heat equation, one gets:

$$\frac{\Delta\varphi(x)}{\varphi(x)} = -\frac{\alpha'(t)}{\alpha(t)} \tag{19}$$

This gives us the following system of ODEs:

$$\begin{cases} \alpha' = -\lambda\alpha \\ \Delta\varphi = \lambda\varphi \end{cases}$$

The first ODE is trivial: $\alpha(t) = e^{-\lambda t}$, and so particular solutions would be of the form

$$u_k = e^{-\lambda_k t}\varphi_k \tag{20}$$

The principle of superposition allows one to add up everything to obtain the solution u:

$$u(x,t) = \sum_k a_k e^{-\lambda_k t}\varphi_k(x) \tag{21}$$

where $a_1, a_2, ... \in \mathbb{R}$ depend on the initial and boundary conditions. To fully solve the PDE, one needs to understands the eigenvalues and eigenvectors (the second ODE). Depending on the geometry of the domain (e.g., interval, sphere, torus, etc.), this can be a highly non-trivial task.

We want to in particular understand the eigenfunctions of the Laplacian; these objects will allow us to work on "Fourier Series" on Riemannian manifolds – spherical harmonics on the n-sphere in our case. The following theorem is essential to the justification for the existence of such objects:

Theorem 3 (Sturm-Liouville's Decomposition)**.** Let (M, g) be a compact Riemannian manifold. Then, there exists an orthonormal basis $\{\varphi_1, , \varphi_2, ...\}$ of the Laplacian Δ_g with the respective eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$ such that any function $\varphi \in L^2(M)$ can be written as a convergent series in $L^2(M)$ for some coefficients $a_j \in \mathbb{R}$:

$$\varphi = \sum_{j=1}^{\infty} a_j\varphi_j \tag{22}$$

3. Examples of the Spectrum of the Laplacian in Various Spaces

Before deriving the spectrum of the Laplacian on the 2- and 3-sphere, we will find the spectrum of the Laplacian first on an interval $[0, \ell]$, a rectangle $[0, \ell] \times [0, m]$, and then on the unit disc (to build some intuition).

3.1. The Interval

First consider an interval of length ℓ, $\Omega := [0, \ell]$ with Dirichlet boundary conditions ($\varphi|_{\partial\Omega} = 0$). Then, one has that the eigenfunctions of Δ are:

$$\varphi_k(x) = \sin\left(\frac{k\pi}{\ell}x\right) \tag{23}$$

where $k \geq 1$ with corresponding eigenvalues:

$$\lambda_k = \left(\frac{k\pi}{\ell}\right)^2 \tag{24}$$

3.2. The Rectangle

Now consider a rectangle $\Omega = [0, \ell] \times [0, m]$. Separating variables with the Ansatz $\varphi(x, y) = f(x)g(y)$, for Dirichlet boundary conditions, one has that the eigenfunctions are:

$$\varphi_{jk}(x, y) = \sin\left(\frac{j\pi}{\ell}\right)\sin\left(\frac{k\pi}{m}y\right) \tag{25}$$

with corresponding eigenvalues

$$\lambda_{jk} = \left(\frac{j\pi}{\ell}\right)^2 + \left(\frac{k\pi}{m}\right)^2 \tag{26}$$

for $j, k \geq 1$.

3.3. The unit disc

For the unit disk, we'll instead study the Laplacian in polar coordinates. This is given by:

$$\Delta := -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{27}$$

Making the Ansatz $\varphi(r, \theta) = R(r)\Phi(\theta)$, one is led to **Bessel's Equation**:

$$x^2 J''(x) + xJ'(x) + (x^2 - k^2)J(x) = 0 \tag{28}$$

where $x = \sqrt{\lambda}r$ and $J(x) = R(x/\sqrt{\lambda})$. The solution to Equation (28) is given by the kth **Bessel Function**:

$$J_k(x) = \sum_{k=0}^{\infty} \frac{(-1)^\ell}{\ell!(k+\ell)!}\left(\frac{x}{2}\right)^{k+2\ell} \tag{29}$$

which gives us that

$$\varphi_k^\lambda = \Phi_k(\theta)J_k(\sqrt{\lambda}/r) \tag{30}$$

are the eigenfunctions of Δ, where

$$\Phi_k(\theta) = a_k\cos(k\theta) + b_k\sin(k\theta) \tag{31}$$

3.4. The Circle

Finally, we'll derive the spectrum of the Laplacian on the circle. The circle is defined by $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. The eigenfunctions of Δ can be found by trying to find the eigenfunctions φ of $-\frac{d^2}{dt^2}$ on $\mathbb{R}$, where φ satisfies the following periodicity requirement:

$$\varphi(t) = \varphi(t + 2\pi j) \tag{32}$$

for $j \in \mathbb{Z}$, $t \in \mathbb{R}$. Thus, the eigenfunctions are exactly what one would expect based on the standard Fourier series one would see in a course on PDEs:

$$1, \cos(t), \sin(t), \cos(2t), \sin(2t), ..., \cos(kt), \sin(kt), ... \tag{33}$$

with eigenvalues of $0, 1, 1, 4, , 4, ..., k^2, k^2, ...$ (respectively).

4. Spherical Harmonics

4.1. Eigenfunctions of Δ_g on the n-sphere: theoretical

The first task to do is to write the Euclidean metric on $\mathbb{R}^{n+1}$ in terms of the round metric on S^n. We obtain:

$$g_{\mathbb{R}^{n+1}}(r, \xi) = \begin{bmatrix} 1 & 0 \\ 0 & r^2 g_{S^n}(\xi) \end{bmatrix} \tag{34}$$

This gives us the following expression for the Laplacian with respect to the metric $g_{\mathbb{R}^{n+1}}$:

$$\Delta_{g_{\mathbb{R}^{n+1}}} = -\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_{g_{S^n}} \tag{35}$$

A **homogeneous polynomial** is a polynomial whose non-zero terms all have the same degree. For example, $p(x, y) = x^4 + x^2y^2 + 7y^4$ is a homogeneous polynomial whereas $q(x, y) = x^4 + x^3 + yx^2$ is a non-homogeneous polynomial. Define the following sets:

$$\mathcal{P}_k := \{ \text{ homogeneous polynomials of degree } k \ \} \tag{36}$$
$$\mathcal{H}_k := \{P \in \mathcal{P}_k \mid g_{\mathbb{R}^{n+1}} P = 0\} \tag{37}$$
$$H_k := \{P|_{S^n} \mid P \in \mathcal{H}_k\} \tag{38}$$

That, is, $\mathcal{H}_k$ is the set of all harmonic polynomials of degree k with respect to the Laplacian with the metric $g_{\mathbb{R}^{n+1}}$. H_k is the restriction of such polynomials to the n-sphere. There is a natural relation between $\mathcal{H}_k$ and H_k:

Proposition 1. $H_k \simeq \mathcal{H}_k$. Moreover,

$$H_k \subseteq \{Y \in C^\infty(S^n) \mid \Delta_{g_{S^n}} Y = k(n + k - 1)Y\} \tag{39}$$

That is, if $Y \in H_k$, then Y is an eigenfunction of $\Delta_{g_{S^n}}$ with eigenvalue $k(n + k - 1)$.

Proof. To prove that they are isomorphic, we will first prove that the restriction map given by

$$\mathcal{H}_k \to H_k$$
$$P \mapsto P|_{S^n}$$

is invertible, with the following inverse:

$$\begin{aligned} H_k &\to \mathcal{H}_k \\ Y &\mapsto r^k Y \end{aligned}$$

To that end, let $Y \in H_k$ be arbitrary. Then, by definition $Y = P|_{S^n}$ for some harmonic polynomial $P \in \mathcal{H}_k$.We obtain:

$$P(x) = P\left(||x||\frac{x}{||x||}\right) \underbrace{=}_{(1)} ||x||^k P\left(\frac{x}{||x||}\right) \tag{40}$$

where the equality (1) follows from the fact that P is a homogeneous polynomial. However, since $x/||x|| \in S^n$, we can write:

$$P(x) = ||x||^k P\left(\frac{x}{||x||}\right) = ||x||^k Y(x/||x||) \tag{41}$$

and so we have that $Y \mapsto r^k Y$. Now all that remains to show is that for a harmonic polynomial restricted to the n-sphere $Y \in H_k$, one has that Y is an eigenfunction of $\Delta_{g_{S^n}}$. To that end, let $Y \in H_k$ be arbitrary. Then, $\exists$ a $P(x) \in \mathcal{H}_k$ such that $Y = P|_{S^n}$. By the homogeneity of P, one gets that $P(x, \xi) = r^k Y(\xi)$. Now, P is harmonic, and so we can expand out $\Delta g_{\mathbb{R}^{n+1}} P$ as:

$$\begin{aligned} 0 &= \Delta g_{\mathbb{R}^{n+1}} P \\ &= \Delta g_{\mathbb{R}^{n+1}} r^k Y(\xi) \\ &= -Y \frac{1}{r^n} \frac{\partial}{\partial r}\left[k r^{n+k-1}\right] + \frac{1}{r^2} r^k \Delta g_{S^n} Y \\ &= -Y \frac{1}{r^n} \frac{\partial}{\partial r}\left[k r^{n+k-1}\right] + r^{k-2} \Delta g_{S^n} Y \\ &= \frac{-k}{r^n}[n+k-1] r^{n+k-2} Y + r^{k-2} \Delta g_{S^n} Y \\ &= -k(n+k-1) r^{k-2} Y + r^{k-2} \Delta g_{S^n} Y \\ &= r^{k-2}[\Delta g_{S^n} Y - k(n+k-1) Y] \end{aligned}$$

Since $r^{k-2} \neq 0$, $\Delta g_{S^n} Y - (n+k-1)Y = 0$ which occurs $\iff$ $\Delta g_{S^n} Y = k(n+k-1)Y$ $\iff$ Y is an eigenfunction of Δg_{S^n} with eigenvalue $k(n+k-1)$, which is what we wanted to show. ☐

Proposition 2. Set $r^2 \mathcal{P}_{k-2} := \{r^2 P \mid P \in \mathcal{P}_{k-2}\}$. Then, one has:

$$\mathcal{P}_k = \mathcal{H}_k \oplus \mathcal{P}_{k-2} \tag{42}$$

From Proposition (2), one can obtain a finer decomposition:

Corollary 1. One can write:

$$\mathcal{P}_{2k} = \mathcal{H}_{2k} \oplus r^2 \mathcal{H}_{2k-2} \oplus r^4 \mathcal{H}_{2k-4} \oplus \cdots \oplus r^{2k} \mathcal{H}_{\prime} \tag{43}$$

$$\mathcal{P}_{2k+1} = \mathcal{H}_{2k+1} \oplus r^2 \mathcal{H}_{2k-1} \oplus r^4 \mathcal{H}_{2k-3} \oplus \cdots \oplus r^{2k} \mathcal{H}_1 \tag{44}$$

Corollary 1 immediately tells us that we can write any homogeneous harmonic polynomial restricted to the sphere as the sum of spherical harmonics. More precisely, one as

Corollary 2. Let $P \in \mathcal{P}_k$.Then, the restriction of P to the n-sphere, $P|_{S^n}$, is the sum of spherical harmonics of degree $\leq k$.

One can use a combinatorial argument and the fact that $H_k \simeq \mathcal{H}_k$, one can prove the following dimension formula:

Corollary 3.

$$\dim(H_k) = (2k+n-1)\frac{(k+n-2)!}{k!(n-1)!} \tag{45}$$

We also have the following dimension formula for the dimension of $\mathcal{P}_k$:

$$\dim(P_k) = \binom{n+k}{n} \tag{46}$$

We can now state and prove the most important theorem of this section: the eigenfunctions of the Laplacian on the n-sphere embedded in $\mathbb{R}^n$ ($S^n \subseteq \mathbb{R}^n$) are the restrictions of the harmonic polynomials to the sphere. The corresponding eigenspaces are H_k with eigenvalues $\lambda_k = k(k+n-1)$ with multiplicities $(2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}$.

Theorem 4. We have the following decomposition of $L^2(S^n)$ in terms of the spherical harmonics:

$$L^2(S^n) = \bigoplus_{k=1}^{\infty} H_k \tag{47}$$

Proof. "$\supseteq$": this inclusion is clear, since it's clear that harmonic polynomials are in $L^2(S^n)$.
"$\subseteq$": By definition, one has that

$$\bigoplus_{k=1}^{\infty} H_k = \bigoplus_{k=1}^{\infty} \mathcal{H}_k|_{S^n} \tag{48}$$

Taking $r = 1$ and using that for $Y \in H_k$, Y can be expressed as the sum of harmonic polynomials restricted to the sphere, one has the following equality:

$$\bigoplus_{k=1}^{\infty} \mathcal{H}_k|_{S^n} = \bigoplus_{k=1}^{\infty} \mathcal{P}_k|_{S^n} \tag{49}$$

We will now prove that $\bigoplus_{k=1}^{\infty} \mathcal{P}_k|_{S^n}$ is dense in $L^2(S^n)$. To do so, we want to use the Stone-Weierstrass Approximation theorem. S^n is a topological manifold, and therefore it is Hausdorff. It is also compact. We also clearly have that $\mathcal{P}_k \cdot \mathcal{P}_\ell \subseteq \mathcal{P}_{k+l}$, which gives us that $\bigoplus_{k=1}^{\infty} \mathcal{P}|_{S^n}$ is a sub-algebra of $C^\infty(S^n)$. We also have that $\bigoplus_{k=1}^{\infty} \mathcal{P}|_{S^n}$ separates points. To see why, let $\mathcal{P}_1 \subseteq \bigoplus_{k=1}^{\infty} \mathcal{P}|_{S^n}$and let $y, z \in S^n$, $y := (y_1, ..., y_{n+1})$, $z := (z_1, ..., z_{n+1})$ such that $y \neq z$. Then, there exists an index $j \in \{1, ..., n+1\}$ for which $y_j \neq z_j$. Choose $P \in \mathcal{P}_1$ such that $P(x_1, ..., x_{n+1}) := x_j$, which gives us that $P(z) \neq P(y)$. We can thus apply the Stone-Weierstrass Theorem to conclude that $\bigoplus_{k=1}^{\infty} H_k$ is dense in $L^2(S^n)$, and so we are done. □

4.2. Eigenfunctions of Δ_g on the 2-sphere

Since $n = 2$, applying the above formulae gives us that the eigenvalues are $\lambda_k = k(k+1)$ with multiplicities $2k+1$,where $k \in \mathbb{N}$. For intuition-building, we'll first find some of the lower-degree spherical harmonics before deriving the full solution. First, we parameterise S^2 by:

$$(\theta, \varphi) \mapsto (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)) \tag{50}$$

$k = 1$: Observe that $\mathcal{P}_1 = \text{span}\{x, y, z\}$. Using the dimension formula, one has that

$$\dim(\mathcal{P}_1) = \binom{2+1}{2} = 3 \tag{51}$$

The harmonic polynomials $\mathcal{H}_1$ is spanned by $\{x, y, z\}$. Re-writing $\mathcal{H}_1 = \text{span}\{x, y, z\}$ in terms of the parametrisation (50), one gets:

$$H_1 = \text{span}\{\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)\} \tag{52}$$

The eigenspace H_1 corresponds to the eigenvalue $\lambda = 1(1+1) = 2$ with multiplicity 3.

$k = 2$: observe that $\mathcal{P}_2 = \text{span}\{x^2, y^2, z^2, xy, xz, yz\}$. Using the dimension formula,

$$\dim \mathcal{P}_2 = \binom{2+2}{2} = 6 \tag{53}$$

the Harmonic polynomials are spanned by $\mathcal{H}_2 = \text{span}\{z^2 - x^2, z^2 - y^2, xy, xz, yz\}$. Restricting $\mathcal{H}_2$ to the sphere, one gets:

$$\begin{aligned} H_2 = \text{span}\{&\cos^2(\theta) - \sin^2(\theta)\cos^2(\varphi), \cos^2(\theta) - \sin^2(\theta)\cos^2(\varphi), \sin^2(\theta)\cos(\varphi)\sin(\varphi), \\ &\sin(\theta)\cos(\varphi)\cos(\theta), \sin(\theta)\sin(\varphi)\cos(\theta)\} \end{aligned}$$

the eigenvalue corresponding to the eigenspace H_2 is $\lambda_2 = 6$, which has multiplicity 5.

We will now find a basis for H_k for any k, which will give us the spherical harmonics on the 2-sphere. To do so, one needs to solve Laplace's Equation $\Delta\varphi = 0$. Set $\varphi(x, y, z) = \psi(r, \theta, \varphi)$, and make the following Ansatz:

$$\psi(r, \theta, \varphi) = R(r) \cdot Y(\theta, \varphi) \tag{54}$$

where $R(r)$ is the radial part and $Y(\theta, \varphi)$ is the angular part. Inserting (54) into Laplace's equation gives us the following eigenvalue problem:

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial R(r)}{\partial r}\right] = \ell(\ell+1)R(r) \tag{55}$$

$$\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left[\sin(\theta)\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right] + \frac{1}{\sin^2(\theta)}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2} = -\ell(\ell+1)Y(\theta,\varphi) \tag{56}$$

for some separation constant $\ell(\ell+1)$. We'll apply separation of variables once more to the bottom equation. Make the following Ansatz:

$$Y(\theta, \varphi) = \Theta(\theta)e^{im\varphi} \tag{57}$$

For some separation constant $m \in \mathbb{Z}$. Substituting $Y(\theta, \varphi) = \Theta(\theta)e^{im\varphi}$ into the bottom equation, one obtains a quite famous ODE:

$$\sin(\theta)\frac{\partial}{\partial\theta}\left[\sin(\theta)\frac{\partial\Theta(\theta)}{\partial\theta}\right] = m^2\Theta(\theta) - \ell(\ell+1)\sin^2(\theta)\Theta(\theta) \tag{58}$$

The solutions to (58) are given by the **Legendre Polynomials**:

$$P_\ell^m(x) := \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2+1)^\ell \tag{59}$$

where $\ell > 0, m \in \mathbb{Z}$, $|m| < \ell$. The solutions for $Y(\theta, \varphi)$ are thus expressed in terms of the Legendre polynomials. This representation is called the **spherical harmonics** and are given by:

$$Y_\ell^m(\theta,\varphi) = \underbrace{\sqrt{\frac{(2\ell+1)(k-|m|)!}{4\pi(\ell+m)}}}_{:=C_\ell^m} Y_\ell^m(\cos(\theta))e^{im\varphi} \tag{60}$$

where the C_ℓ^m are the normalising constants such that $||Y_\ell^m||_2 = 1$. We call m the **degree** of the solution and ℓ the **order** of the solution.

We have that any $f \in L^2(S^2)$ can be expressed as a linear combination of spherical harmonics:

$$f(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\theta,\varphi) \tag{61}$$

We obtain the coefficients f_m^ℓ in an analogous way to the Fourier coefficients by taking inner products:

$$\begin{aligned} f_\ell^m &= \int_{S^2} f(\theta,\varphi)[Y_\ell^m(\theta,\varphi)]^* d\Omega \\ &= \int_0^{2\pi} d\varphi \int_0^\pi f(\theta,\varphi)[Y_\ell^m(\theta,\varphi)]^* \sin(\theta) d\theta \end{aligned}$$

where $[Y_\ell^m(\theta,\varphi)]^*$ is the complex conjugate of $Y_\ell^m(\theta,\varphi)$.

4.3. Eigenfunctions of Δ_g on the 3-sphere

Analogously, we can study the spectrum of the Laplacian on the 3-sphere, and in doing so obtain a formula for the spherical harmonics on the 3-sphere. We will let $Y^{\ell k m}$ denote the spherical harmonics on S^3, and therefore $k, \ell, m \in \mathbb{Z}$ will denote the order of the harmonic. We require that $-\ell \leq m \leq \ell$. The spherical harmonics $Y^{k\ell m}$ are the eigenfunctions of the covariant Laplace operator, and so we want to solve the following eigenvalue problem:

$$\nabla^a\nabla_a Y^{k\ell m} = -\frac{k(k+2)}{R_3^2} Y^{k\ell m} \tag{62}$$

where R_3 is the radius of the 3-sphere and ∇_a is the covariant derivative with respect to the round metric on S^3. Parameterise the 3-sphere as $x^a = (\chi, \theta, \varphi)$ and the metric becomes:

$$\begin{aligned} ds^2 &= g_{ab} dx^a dx^b \\ &= R_3^2[d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2(\theta)d\varphi^2)] \end{aligned}$$

We'll use the 2-sphere spherical harmonics to obtain an expression for the 3-sphere spherical harmonics $Y^{\ell mn}$ by defining functions $H^{k\ell}(\chi)$ that we'll derive:

$$Y^{k\ell m}(\chi,\theta,\varphi) = H^{k\ell}(\chi) Y^{\ell m}(\theta,\varphi) \tag{63}$$

We'll derive $H^{k\ell}(\chi)$ by inserting (63) into the eigenvalue problem (62). This gives us the following ODE:

$$0 = \frac{d^2H^{k\ell}}{d\chi^2} + 2\cot(\chi)\frac{dH^{k\ell}}{d\chi} + [k(k+2) - \ell(\ell+1)\csc^2(\chi)]H^{k\ell} \tag{64}$$

which can be solved by introducing functions $C^{k\ell}(\chi)$ such that

$$H^{k\ell}(\chi) = \sin^\ell(\chi) C^{k\ell}(\chi) \tag{65}$$

For $k = \ell$ and $k = \ell + 1$, the $C^{k\ell}$ are given by the following expressions:

$$C^{\ell\ell} = (-1)^{\ell+1} 2^\ell \ell! \sqrt{\frac{2(\ell+1)}{\pi(2\ell+1)!}} \tag{66}$$

$$C^{(\ell+1)\ell} = \sqrt{2(\ell+2)}\cos(\chi) C^{\ell\ell} \tag{67}$$

and for $k > \ell + 1$, we have the following recursive formula:

$$C^{k+2\ell} = 2\cos(\chi)\sqrt{\frac{(k+3)(k+2)}{(k+3+\ell)(k+2-\ell)}} C^{(k+1)\ell} - \sqrt{\frac{(k+3)(k+2+\ell)(k+1-\ell)}{(k+1)(k+3+\ell)(k+2-\ell)}} C^{k\ell} \tag{68}$$

As a sanity check, the spherical harmonics defined by

$$Y^{k\ell m}(\chi) = \sin^\ell(\chi) C^{k\ell}(\chi) Y^{\ell m}(\theta, \varphi) \tag{69}$$

should satisfy the following orthogonality condition:

$$\delta^{kk'}\delta^{\ell\ell'}\delta^{mm'} = \frac{1}{R_3^3}\int_{S^3} Y^{k\ell m}[Y^{k\ell m}]^* \sqrt{g} d^3x \tag{70}$$

where δ^{ij} is the Kronecker delta function

$$\delta^{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(which they do). And, similar to what we did with Fourier series and spherical harmonics on the 2-sphere, any scalar function $f : S^3 \to \mathbb{R}$ can be expressed in terms of the spherical harmonics:

$$f = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=-\ell}^{\ell} S^{k\ell m} Y^{k\ell m} \tag{71}$$

where the coefficients $S^{k\ell m}$ can again be found by taking inner products (integrating on the 3-sphere):

$$S^{k\ell m} = \frac{1}{R_3^3}\int_{S^3} f [Y^{k\ell m}]^* \sqrt{g} d^3x \tag{72}$$

5. Summary Tables for Quick Reference

S^1: the Fourier series of $f(x)$ on $[-L, L]$ is given by:

$$FS[f(x)] := \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{73}$$

where the Fourier coefficients can be found by taking inner products:

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = \frac{\langle f, \cos\left(\frac{n\pi x}{L}\right)\rangle}{\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\rangle} \tag{74}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{\langle f, \sin\left(\frac{n\pi x}{L}\right)\rangle}{\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\rangle} \tag{75}$$

$$\frac{1}{2}A_0 = \int_{-L}^{L} f(x) dx = \langle f(x), 1\rangle \tag{76}$$

S^2: Let $f \in L^2(S^2)$. Then:

$$f(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\theta,\varphi) \tag{77}$$

where

$$f_\ell^m = \int_0^{2\pi} d\varphi \int_0^{\pi} f(\theta,\varphi)[Y_\ell^m(\theta,\varphi)]^* \sin(\theta) d\varphi \tag{78}$$

where,

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{(2\ell+1)(k-|m|)!}{4\pi(\ell+m)!}} \tag{79}$$

where

$$P_\ell^m(\cos(\theta)) = \frac{(-1)^m}{2^\ell \ell!}[1-\cos^2(\theta)]^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}}(\cos^2(x)+1)^\ell \tag{80}$$

Spherical harmonics on S^2 can be looked up, see the chart below for some of them:

List of spherical harmonics [edit]

Main article: Table of spherical harmonics

Analytic expressions for the first few orthonormalized Laplace spherical harmonics that use the Condon-Shortley phase convention:

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta\, e^{-i\varphi}$$

$$Y_1^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta$$

$$Y_1^1(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta\, e^{i\varphi}$$

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{-2i\varphi}$$

$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \sin\theta\, \cos\theta\, e^{-i\varphi}$$

$$Y_2^0(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}} (3\cos^2\theta - 1)$$

$$Y_2^1(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{15}{2\pi}} \sin\theta\, \cos\theta\, e^{i\varphi}$$

$$Y_2^2(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{2i\varphi}$$

Figure 1: Some spherical harmonics

S^3: Let $f \in L^2(S^3)$. Then,

$$f = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=-\ell}^{\ell} S^{k\ell m} Y^{k\ell m} \tag{81}$$

where,

$$Y^{k\ell m}(\chi,\theta,\varphi) = \sin^\ell(\chi) C^{k\ell}(\chi) Y^{\ell m}(\theta,\varphi) \tag{82}$$

where $Y^{\ell m}(\theta, \varphi)$ are spherical harmonics on S^2. We have that for $k = \ell$ and $k = \ell + 1$,

$$C^{\ell\ell} = (-1)^{\ell+1} e^{\ell} \ell! \sqrt{\frac{2(\ell+1)}{\pi(2\ell+1)}} \tag{83}$$

$$C^{(\ell+1)\ell} = \sqrt{2(\ell+2)} \cos(\chi) C^{\ell\ell} \tag{84}$$

and for $k > \ell + 1$,

$$C^{(k+2)\ell} 2\cos(\chi) \sqrt{\frac{(k+3)(k+2)}{(k+3-\ell)(k+2-\ell)}} C^{(k+1)\ell} - \sqrt{\frac{(k+3)(k+2+\ell)(k+1-\ell)}{(k+1)(k+3+\ell)(k+2-\ell)}} C^{k\ell} \tag{85}$$

Finally, the coefficients $S^{k\ell m}$ can be calculated by inner products

$$S^{k\ell m} \frac{1}{R_3^3} \int_{S^3} f [Y^{k\ell m}]^* \sqrt{g} d^3x \tag{86}$$

6. Applications of Spherical Harmonics

To be done, at some later date, certainly not now.

6.1. Exoplanet Mapping

6.2. Quantum Mechanics: Hydrogen Atom

6.3. Classical Electrodynamics

6.4. Schwarzschild black hole

7. References

(1) "Analysis of Laplacian on manifolds" course notes

(2) Math 475 Lecture Notes

(3) "Scalar, Vector and Tensor Harmonics on the Three-Sphere"

(4) "Real Analysis" by Royden & Fitzpatrick

(5) `https://en.wikipedia.org/wiki/Spherical_harmonics`